Tensor

Before moving to the details of tensor, we review here some elementary physical laws which will give you some feelings of tensor uses in physics.

$$\vec{a} = \frac{\vec{F}}{m} \quad (1)$$ → acceleration of a body is proportional to the force acting on it

$$\vec{J} = \sigma \vec{E} \quad (2)$$ → Electric ~~field~~ current in a medium is proportional to the applied field.

The above physical laws are a special case and apply strictly only to isotropic media (a medium whose properties are same in all directions) or a media which posses high symmetry.

In case of anisotropic media acceleration $\vec{a}$ is not necessarily parallel to the applied force ~~and~~ (Eq-1) Or the current flows in a direction different from that of the electric field (Eq. 2).

Eq-1 or Eq. 2 for anisotropic media can be written in a generalized form. We take Eq 2

$$J_x = \sigma_{xx} E_x + \sigma_{xy} E_y + \sigma_{xz} E_z$$
$$J_y = \sigma_{yx} E_x + \sigma_{yy} E_y + \sigma_{yz} E_z$$
$$J_z = \sigma_{zx} E_x + \sigma_{zy} E_y + \sigma_{zz} E_z$$

J_x, J_y, J_z and E_x, E_y, E_z are respectively the Cartesian components of $\vec{J}$ and $\vec{E}$, and σ_{ij} ($i, j = x, y, z$) are said to be the components of the conductivity tensor

Similar Eq ① can be generalized with $\left(\frac{1}{m}\right)_{ij}$ denoting the components of mass tensor (reciprocal mass tensor) of the particle in the medium.

Tensor Application — mainly in relativistic physics → special theory of relativity, general theory of relativity etc

Conventions & Notations:

Consider an N-dimensional space and let $x^1, x^2, x^3, \ldots, x^N$ be any set of coordinate in this space.

Note that here in writing x^i, i is the superscript on x not the ith power on x.
When it will be needed to write power on x^i we will write it like $(x^i)^2$, $(x^i)^3$, etc

N-dimensional space under the consideration will be denoted by V_n.

A notation $f \equiv f(t)$ will mean that f is a function of t.

Let $\bar{x}^\alpha$ ($1 \leq \alpha \leq N$) be another set of coordinates in the same space V_n. Each of the coordinates x^i will be a function of the N coordinates $\bar{x}^\alpha$, and vice versa. Therefore, we can write

$$x^i = x^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N), \qquad 1 \leq i \leq N \quad —— ③$$

$$\bar{x}^\alpha = \bar{x}^\alpha(x^1, x^2, \ldots, x^N), \qquad 1 \leq \alpha \leq N \quad —— ④$$

Example: Express the Cartesian and the Spherical polar coordinates as functions of each other.

Soln: We have already discussed curvilinear coordinates in earlier lecture note. We use ~~some~~ those concepts here.

Let x, y, z denote the Cartesian coordinates and r, θ, ϕ the Spherical polar Coordinates in a three-dimensional Space. The coordinates x, y, z are independent of each other. Similarly, r, θ, ϕ are indebendent of each other.

The coordinates of one set are functions of those of the other set. The Cartesian coordinates are related to the Spherical polar coordinates by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

The inverse transformation is given by

$$r = (x^2+y^2+z^2)^{\frac{1}{2}}, \quad \theta = \tan^{-1}\left[\frac{\sqrt{x^2+y^2}}{z}\right], \quad \phi = \tan^{-1}(y/x)$$

You see that (r, θ, ϕ) can be expressed as functions of (x, y, z), and vice versa.

Now differentiating Equations (3) and (4) we can write.

$$dx^i = \sum_{\alpha=1}^{N} \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha, \quad 1 \leq i \leq N \quad \text{— (6)}$$

$$d\bar{x}^\alpha = \sum_{i=1}^{N} \frac{\partial \bar{x}^\alpha}{\partial x^i} dx^i, \quad 1 \leq \alpha \leq N. \quad \text{(7)}$$

If we use Einstein's Summation convention, it will simplify the above notations.